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Periodic Solutions of a Non-Linear Parabolic Equation

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1. INTRODUCTION

There is a considerable literature associated with diffusion equations

$$\dot{u} = A(u) + f \tag{1}$$

with A a linear or nonlinear elliptic operator. Most of this literature focuses on initial/boundary value problems but several papers (e.g., [3, 9]) have considered the existence of solutions for problems in which the specification of initial data is replaced by the imposition of a requirement that the solution be periodic (in t); A, f, and the boundary data must, of course, share this periodicity.

The "obvious" approach to periodicity would be, as is standard for ordinary differential equations, to use the theory of the initial value problem to define a period map $S: u_0 \mapsto u_1$ (u_0 the initial data and u_1 the resulting solution state a period later) and then to show that S satisfies the hypotheses of some appropriate fixpoint theorem. This is Browder's approach in [3], for example. Here, on the other hand, we employ a static (function of t, x) rather than dynamic ($t \mapsto$ function of x) formulation; such a formulation has been used by Brezis [1], for example, in connection with initial value problems. Thus, rather than looking among solutions for one which is periodic, we seek a solution in some suitable space of t-periodic functions.

We consider a number of questions for such problems in the context of elliptic operators $A(\cdot)$ having the special form

$$A(u) = \nabla \cdot G(\nabla u) \tag{2}$$

where $G: \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$G(X) = \gamma(|X|)X. \tag{3}$$

Our results will depend on the behavior of the scalar function

$$G_0(r) = r\gamma(r), \qquad (r > 0).$$

This special form generalizes, for example, an equation arising in the analysis of eddy currents induced in a homogeneous ferroconductor by an external magnetic field. Under conditions of longitudinal invariance the vector potential has a single nonvanishing component u satisfying the scalar equation: $\dot{u} = \nabla \cdot \gamma \nabla u$. Here γ is the reciprocal of the magnetic permeability of the material.

Usually γ is taken to be constant, making the equation linear, but in some cases of physical significance it must be taken as dependent on the strength of the magnetic field, from which it follows that $\gamma = \gamma(|\nabla u|)$. (For an inhomogeneous material, γ may be a function of position as well.) In this setting one has, on physical grounds, that

$$0 < \gamma(r) < M, \qquad G_0'(r) \geqslant m > 0 \qquad (r > 0) \tag{4}$$

for suitable constants m, M.

If the boundary data (determined by the external field) are periodic in time, we seek a solution with the same periodicity. The author initially considered the problem of numerical computation of this periodic solution (cf. [8]). In the context of numerical solution it is, of course, most important to be assured of not only the existence of a unique solution but of its continuous dependence on the data (including not only the boundary data but also the exact functional form of γ, which is known only through measurement).

2. Some Lemmas

The precise setting in which the results will be obtained will depend on the behavior of the function $G_0(r) = r\gamma(r)$. The original conditions (4) require that γ be "essentially constant" so G_0 is "essentially linear." We will consider the more general situation in which G_0 behaves like a power. More precisely, we require that (for some $\alpha > 0$):

(a) G_0 is positive and continuous on $(0, \infty)$;

(b) there exists $M_0 > 0$ such that for $r > 0$

$$G_0(r) \leqslant M_0(1 + r^\alpha),$$

(c) there exist $m_0 > 0$, $r_0 > 0$ such that for $r \geqslant r_0$ (5)

$$G_0(r) \geqslant m_0 r^\alpha,$$

(d) for each $r > 0$ there exists $m = m(r) > 0$ such that for $r_1 \geqslant r_2 > r$

$$G_0(r_1) - G_0(r_2) \geqslant m(r_1^\alpha - r_2^\alpha).$$

Condition (5c) is, of course, an immediate consequence of (5d) but is stated separately as some of our results do not require the full strength of (5d). Clearly (4) implies (5) with $\alpha = 1$. For continuously differentiable G_0 the condition (5d) is equivalent to

$$G_0' > 0 \text{ on } (0, \infty), \qquad \liminf_{r\to\infty} r^{1-\alpha} G_0'(r) > 0.$$

Given a finite measure space (S, μ), set $p = \alpha + 1$ and consider the Banach space $\mathscr{X} = L_p(S \to \mathbb{R}^n)$. For $z_1, z_2 \in \mathscr{X}$ let

$$\Gamma(z_1)\, z_2 = \int_S G(z_1(s)) \cdot z_2(s)\, \mu(ds). \tag{6}$$

LEMMA 1. *Let $G\colon \mathbb{R}^n \to \mathbb{R}^n$ be defined as in* (3) *with G_0 satisfying the conditions* (5a,b). *Then for each $z_1 \in \mathscr{X}$ the linear functional $\Gamma(z_1)$ is well-defined by* (6) *and continuous in z_2; the map $\Gamma\colon \mathscr{X} \to \mathscr{X}^*$ so defined is continuous and takes bounded sets to bounded sets. If, in addition, G_0 satisfies* (5c), *then Γ is coercive so $\Gamma(z + z_0)z/\| z \| \to \infty$ as $\| z \| \to \infty$.*

Proof. From (5a) one has $G \circ z_1$ measurable and from (5b) one has

$$\begin{aligned}\int_S G(z_1) \cdot z_2 \mu(ds) &\leqslant M_0 \int_S [1 + | z_1 |^{p-1}]\, | z_2 |\, \mu(ds)\\ &\leqslant M_0\{\mu(S)^{1/q} + \| z_1 \|^{p/q}\}\, \| z_2 \|,\end{aligned}$$

$(q = p/(p-1))$ using the Hölder inequality. Thus $\Gamma(z_1)$ is well-defined and in $\mathscr{X}^*$ with

$$\| \Gamma(z_1)\| \leqslant M_0\{\mu(S)^{1/q} + \| z_1 \|^{p/q}\}.$$

A standard argument for Nemytskii operators (cf., e.g., [6]) continues from this point to show that $\Gamma\colon \mathscr{X} \to \mathscr{X}^*$ is continuous. Now, given (5c) as well, set $S' = \{s\colon | z(s) + z_0(s)| \geqslant r_0\}$ so

$$\begin{aligned}\Gamma(z + z_0)z = {}&\int_{S'} G_0(| z + z_0 |)\, | z + z_0 | + \int_{S\setminus S'} G_0(| z + z_0 |)\, | z + z_0 |\\ &- \int_S G(z + z_0) \cdot z_0\\ \geqslant{}& m_0[\| z + z_0 \|^p - r_0{}^p \mu(S)] - M_0[\mu(S)^{1/q} + \| z + z_0 \|^{p/q}]\, \| z_0 \|.\end{aligned}$$

Since $p/q < p$, the dominant term as $\| z \| \to \infty$ is $m_0 \| z + z_0 \|^p$ which is asymptotically the same as $m_0 \| z \|^p$. Thus, $\Gamma(z + z_0)z/\| z \|$ grows like $\| z \|^\alpha$ as $\| z \| \to \infty$.

LEMMA 2. *Let $G: \mathbb{R}^n \to \mathbb{R}^n$ be defined as in* (3) *with G_0 satisfying* (5a,b,d). *For $z, z_0 \in \mathscr{X}$ set*

$$B(z, z_0) = [\Gamma(z) - \Gamma(z_0)](z - z_0)$$
$$= \int_S [G(z(s)) - G(z_0(s))] \cdot [z(s) - z_0(s)] \, \mu(ds).$$

Then $[B(z, z_0)/\| z - z_0 \|] \to 0$ implies $z \to z_0$ (convergence in $\mathscr{X}$) for any $z_0 \in \mathscr{X}$.

Proof. By Lemma 1, the conditions (5a,b) guarantee that $B(z, z_0)$ is well defined and continuous for $z, z_0 \in \mathscr{X}$. At each $s \in S$, we set $\xi \; (= \xi(s)) = \max\{| z(s)|, | z_0(s)|\}$, $\eta = \min\{| z(s)|, | z_0(s)|\}$, $\theta = z(s) \cdot z_0(s)/\xi\eta$ (so $-1 \leqslant \theta \leqslant 1$) and $\delta = | z(s) - z_0(s)|$. A simple calculation shows that

$$\begin{aligned}\beta = \beta(s) &= [G(z(s)) - G(z_0(s))] \cdot [z(s) - z_0(s)]\\ &= \xi G_0(\xi) - \theta\xi G_0(\eta) - \theta\eta G_0(\xi) + \eta G_0(\eta)\\ &= (\xi - \theta\eta)[G_0(\xi) - G_0(\eta)] + (1 - \theta)(\xi + \eta)\, G_0(\eta).\end{aligned}$$

For any $\epsilon_1, \epsilon_2 \geqslant 0$, set

$$S_1 = \{s: \xi \leqslant \epsilon_1\}, \qquad S_2 = \{s: \delta \leqslant \epsilon_2\xi\}, \qquad S_3 = S\backslash(S_1 \cup S_2).$$

On S_1 one has $\delta \leqslant 2\epsilon_1$ so

$$\int_{S_1} \delta^p \mu(ds) \leqslant (2\epsilon_1)^p \, \mu(S).$$

On S_2 one has $\eta \geqslant \xi - \delta \geqslant (1 - \epsilon_2)\xi$ so

$$\begin{aligned}\int_{S_2} \delta^p \mu(ds) &\leqslant \left(\frac{\epsilon_2}{1 - \epsilon_2}\right)^p \int_{S_2} | z_0(s)|^p \, \mu(ds)\\ &\leqslant \left(\frac{\epsilon_2}{1 - \epsilon_2}\right)^p \| z_0 \|^p.\end{aligned}$$

Finally, on S_3 one has $\epsilon_1 \leqslant \xi \leqslant \delta/\epsilon_2$. Set $m = m(\epsilon_1/4)$ from (5d) and consider two cases: $\eta \leqslant \xi/2$ and $\eta > \xi/2$. In the first case one has

$$\begin{aligned}\beta &\geqslant (\xi - \theta\eta)[G_0(\xi) - G_0(\eta)]\\ &\geqslant (\xi/2)[G_0(\xi) - G_0(\xi/2)]\\ &\geqslant (\xi/2)\, m[\xi^\alpha - 2^{-\alpha}\xi^\alpha]\\ &\geqslant m((1 - 2^{1-p})/2^{p+1})\, \delta^p,\end{aligned}$$

since, in any case, $\xi \geqslant \delta/2$ so $\xi^p \geqslant 2^{-p}\delta^p$. In the second case ($\eta > \xi/2$) one has $\eta/2 \geqslant \epsilon_1/4$ so

$$G_0(\eta) \geqslant G_0(\eta) - G_0(\eta/2) \geqslant m(1 - 2^{-\alpha})\,\eta^\alpha,$$
$$G_0(\xi) - G_0(\eta) \geqslant m(\xi^\alpha - \eta^\alpha) = m\alpha\zeta^{\alpha-1}(\xi - \eta)$$

with $\eta \leqslant \zeta \leqslant \xi$. For $\alpha > 0$ we always have $\alpha > 1 - 2^{-\alpha}$ so

$$\beta \geqslant m(1 - 2^{-\alpha})[\zeta^{\alpha-1}(\xi - \theta\eta)(\xi - \eta) + \eta^{\alpha-1}(1 - \theta)(\xi + \eta)\eta].$$

At this point we distinguish the cases: $\alpha \geqslant 1$ and $0 < \alpha < 1$. If $(\alpha - 1) \geqslant 0$, then

$$\zeta^{\alpha-1} \geqslant \eta^{\alpha-1} \geqslant (\delta/4)^{\alpha-1}$$

while if $(\alpha - 1) \leqslant 0$,

$$\eta^{\alpha-1} \geqslant \zeta^{\alpha-1} \geqslant \xi^{\alpha-1} \geqslant (\delta/\epsilon_2)^{\alpha-1}.$$

Thus, observing that

$$\delta^2 = (\xi - \theta\eta)(\xi - \eta) + (1 - \theta)(\xi + \eta)\eta,$$

we have $\beta \geqslant mC_p\delta^p$ everywhere on S_3. Here,

$$C_p = C_p(\epsilon_2) = \begin{cases} (1 - 2^{-\alpha})\min\{2^{-p-1}, 4^{2-p}\}, & \text{for } \ p \geqslant 2 \quad (\alpha \geqslant 1), \\ (1 - 2^{-\alpha})\min\{2^{-p-1}, \epsilon_2^{2-p}\}, & \text{for } \ p < 2 \quad (0 < \alpha < 1), \end{cases}$$

and we recall that $m = m(\epsilon_1/4)$. Thus,

$$\int_{S_3} \delta^p\mu(ds) \leqslant (1/mC_p)\int_{S_3} \beta\mu(ds) \leqslant B(z, z_0)/mC_p\,.$$

Collecting, we have

$$\begin{aligned} \| z - z_0 \|^p = \int_S \delta^p\mu(ds) &\leqslant \int_{S_1} + \int_{S_2} + \int_{S_3} \\ &\leqslant (2\epsilon_1)^p\,\mu(S) + (\epsilon_2/(1 - \epsilon_2))^p\,\| z_0 \|^p + (1/mC_p)\,B(z, z_0) \end{aligned}$$

with m depending on ϵ_1, and C_p (possibly) depending on ϵ_2. If one chooses ϵ_1, ϵ_2 so the first two terms on the right are each less than $\epsilon^p/4$, this fixes m, C_p so one can make the third term less than $(\epsilon^{p-1}/2)\,\| z - z_0 \|$ by requiring that

$$\frac{B(z, z_0)}{\| z - z_0 \|} \leqslant m(\epsilon_1/4)\,C_p(\epsilon_2)\,\epsilon^{p-1}/2. \tag{7}$$

(If there exists $m(0) > 0$ for (5d) then take $\epsilon_1 = 0$, while for $p \geqslant 2$ one may take $\epsilon_2 = 0$.) Thus, setting $\| z - z_0 \| = \omega$, one has $\omega \geqslant 0$ and

$$\Psi(\omega) := \omega^p - (\epsilon^{p-1}/2)\omega - \epsilon^p/2 \leqslant 0.$$

Since $\Psi' > 0$ for $\omega > (1/3p)^{1/(p-1)}\epsilon$ and $\Psi(0) < 0$, there is only one positive root of Ψ—namely, $\omega = \epsilon$—and $\Psi(\omega) > 0$ for $\omega > \epsilon$. Thus, (7) guarantees $\| z - z_0 \| \leqslant \epsilon$.

3. Principal Results

We now are ready to consider the diffusion equation

$$\dot{u} = \nabla \cdot G(\nabla u) \tag{8}$$

for $t \in \mathbb{R}$ and x in a bounded region Ω in $\mathbb{R}^n$ with, e.g., C^∞ boundary $\partial\Omega$. We impose, for example, Dirichlet conditions

$$u = \varphi \text{ on } \mathbb{R} \times \partial\Omega \tag{9}$$

with φ periodic in t (with no loss of generality, assume the period is 1) and seek a solution of (8), (9) having the same periodicity.

Introduce $\mathbf{T} = \mathbb{R}/\mathbb{Z}$ (i.e., $\mathbb{R}$ with t, t' identified if $(t - t')$ is an integer). The periodicity of φ in t just permits us to consider it as defined on $\mathbf{T} \times \partial\Omega$ rather than on $\mathbb{R} \times \partial\Omega$. Suppose, now, $\hat{\varphi}$ is an extension of φ to $S = \mathbf{T} \times \Omega$. Formally, if we can find a function v defined on S which satisfies

$$\begin{aligned} &\text{(a)} \quad \dot{v} = \nabla \cdot G(\nabla v + \Phi) - f, &&\text{on} \quad S = \mathbf{T} \times \Omega, \\ &\text{(b)} \quad v = 0, &&\text{on} \quad \partial S = \mathbf{T} \times \partial\Omega, \end{aligned} \tag{10}$$

where $\Phi = \nabla\hat{\varphi}$, $f = \hat{\varphi}^{\cdot}$, then $u = v + \hat{\varphi}$ is precisely the periodic solution of (8), (9) being sought.

Suppose G is given by (3) with G_0 nondecreasing and satisfying the conditions (5a,b,c). Consider the Sobolev-type norm

$$\| v \| = \left[\int_S (| v |^p + | \nabla v |^p) \right]^{1/p} \tag{11}$$

with $p = \alpha + 1$ (note that no t-derivatives appear) and let $\mathscr{V}$ be the completion with respect to this norm of $C^\infty(\bar{S})$ and $\mathscr{V}_0$ the closure in $\mathscr{V}$ of $C_0^\infty(S)$. We interpret the condition (10b) as holding in the weak sense that $v \in \mathscr{V}_0$ and will introduce, in a standard fashion, a weak interpretation of the differential equation (10a).

The linear operator $L_0 = \partial/\partial t$ on $C_0^\infty(S)$ to $\mathscr{V}_0^*$ (given by setting $L_0 v\colon w \mapsto \int_S \dot{v}w$ for $w \in \mathscr{V}_0$) has a closed skew-adjoint extension $L\colon \mathscr{D} \to \mathscr{V}_0^*$ with $\mathscr{D} \supset C_0^\infty(S)$ dense in $\mathscr{V}_0$; the formal skew-adjointness of L_0 follows from

$$\begin{aligned}[L_0 v]w + [L_0 w]v &= \int_S (\dot{v}w + \dot{w}v)\\ &= \int_0^1 \int_\Omega (vw)^{\cdot}\, dt\, dx\\ &= 0 \qquad \text{by periodicity.}\end{aligned}$$

For any $v \in \mathscr{V}$ define a linear functional $A(v)$ by setting

$$A(v)\colon w \mapsto \int_S [G(\nabla v + \Phi) \cdot \nabla w + fw] \qquad \text{for} \quad w \in \mathscr{V}_0. \tag{12}$$

We make the assumptions

$$\Phi \in \mathscr{X} = L_p(S \to \mathbb{R}^n), \qquad f \in L_q(S) \tag{13}$$

with $q = p/(p-1)$.

LEMMA 3. *Let G be given by* (3) *with G_0 satisfying* (5a,b) *and let Φ, f satisfy* (13). *Then $A(v)$ is well-defined by* (12) *and continuous in $w \in \mathscr{V}_0$; the map $A\colon \mathscr{V} \to \mathscr{V}_0^*$ so defined is continuous. If, in addition, G_0 is nondecreasing, then A_0, the restriction of A to $\mathscr{V}_0$, is a monotone operator; if G_0 also satisfies* (5c), *then A_0 is coercive and $T = L + A_0$ is a coercive monotone operator from its domain $\mathscr{D}$ to $\mathscr{V}_0^*$.*

Proof. Given (13) we have $[\nabla v + \Phi] \in \mathscr{X}$ for each $v \in \mathscr{V}$; the definition (11) of the norm on $\mathscr{V} \supset \mathscr{V}_0$ also means that $\nabla w \in \mathscr{X}$, $w \in L_p(S)$ for each $w \in \mathscr{V}_0$. Now $A(v)w = \Gamma(\nabla v + \Phi) \cdot \nabla w + \hat{f}w$ (where $\hat{f} \in \mathscr{V}_0^*$ is given by: $\hat{f}w = \int_S fw$ for $w \in \mathscr{V}_0$). By Lemma 1, the map $A\colon \mathscr{V} \to \mathscr{V}_0^*$ is well defined and continuous. It is clear from the identity

$$\beta = (\xi - \theta\eta)[G_0(\xi) - G_0(\eta)] + (1 - \theta)(\xi + \eta)\, G_0(\eta)$$

in the proof of Lemma 2, that $\beta(s) \geqslant 0$ if G_0 is nondecreasing so

$$[A(v_1) - A(v_2)](v_1 - v_2) = B(\nabla v_1, \nabla v_2) = \int_S \beta \geqslant 0$$

and A is monotone. Given (5c), one notes that

$$A(v)v = \Gamma(\nabla v + \Phi)\, \nabla v + \hat{f}v.$$

By the Poincaré Lemma, the p-norm of v is dominated by the p-norm of $|\nabla v|$ for v vanishing on ∂S since Ω was assumed bounded; thus, the $\mathscr{X}$-norm of ∇v is equivalent to the norm (11) for $v \in \mathscr{V}_0$. Applying Lemma 1 shows that $\Gamma(\nabla v + \Phi)\nabla v$ grows like $\| v \|^p$ so $A(v)v/\| v \|$ grows like $\| v \|^{p-1}$ as $\| v \| \to \infty$. Finally, for $z \in C_0^\infty(S)$ one has

$$(Lz)z = (L_0 z)z = \int_S \dot{z} z \mu(ds)$$

$$= \tfrac{1}{2} \int_0^1 \int_\Omega (z^2)^{\cdot}\, dt\, dx = 0$$

by periodicity. Thus, as L is linear and closed,

$$[Lz_1 - Lz_2](z_1 - z_2) = 0 \qquad \text{for} \quad z_1, z_2 \in \mathscr{D},$$

whence $L + A_0$ is monotone [coercive] whenever A_0 is.

For "smooth" functions v, Φ, f, one may multiply (10a) by any $w \in C_0^\infty(S)$ and integrate over S to obtain

$$0 = \int_S [\dot{v}w - (\nabla \cdot G(\nabla v + \Phi))w + fw]\, \mu(ds)$$

$$= \int_S [\dot{v}w + G(\nabla v + \Phi) \cdot \nabla w + fw]\, \mu(ds) = (L + A)(v)w$$

on using the divergence theorem in Ω for the second term, noting that $w = 0$ on $\partial\Omega$. By a weak periodic solution of (10a) we mean an element v of $\mathscr{V}$ which is in the domain $\mathscr{D}$ of L and for which

$$Lv + A(v) = 0. \tag{14}$$

A weak solution of (10) is then an element $v \in \mathscr{D} \cap \mathscr{V}_0$ satisfying (14). The existence of such an element will follow from a result of Browder's (cf., e.g., [2 or 5]) which is here stated in somewhat restricted form.

THEOREM (F. Browder). *Let $\mathscr{B}$ be a reflexive Banach space and $A_* = L + A: \mathscr{D} \to \mathscr{B}^*$ with L a closed linear operator with dense domain $\mathscr{D} = \mathscr{D}(L) = \mathscr{D}(L^*)$. If A is demicontinuous on $\mathscr{B}$ and A_* is monotone and coercive from $\mathscr{D}$ to $\mathscr{B}^*$, then there is some $b \in \mathscr{D}$ with $A_*(b) = 0$.*

Using this, we may now apply the lemmas already obtained to consideration of the problem (10).

THEOREM 1. *Let G be given as in* (3) *with G_0 nondecreasing and satisfying* *the conditions* (5a,b,c); *set* $p = \alpha + 1$ *and* $q = p/\alpha$ (*so* $1/p + 1/q = 1$). *Let* Ω *be a bounded region in* $\mathbb{R}^n$ *with smooth boundary* $\partial\Omega$; *set* $S = \mathbf{T} \times \Omega$ *and* $\mathscr{X} = L_p(S \to \mathbb{R}^n)$ *and let* $\mathscr{V}_0$ *be the completion of* $C_0^\infty(S)$ *in the norm* (11). *Let* L (*with domain* $\mathscr{D} \subset \mathscr{V}_0$) *be the skew-adjoint closure of* $\partial/\partial t$: $C_0^\infty \to \mathscr{V}_0^*$. *Then for every* $\Phi \in \mathscr{X}$, $f \in L_q(S)$ *there exists a weak solution* v *of* (10)—*i.e., an element* $v \in \mathscr{D} \subset \mathscr{V}_0$ *such that*

$$\int_S [\dot{v}w + G(\nabla v + \Phi) \cdot \nabla w + fw]\, \mu(ds) = 0, \qquad \text{for} \quad w \in \mathscr{V}_0. \quad (14')$$

If, in addition, G_0 *is strictly increasing, then this solution is unique and, if* G_0 *satisfies* (5d) *as well, then the solution depends continuously on* $(\Phi, f) \in \mathscr{X} \times L_p(S)$ *in the norm given by* (11).

Proof. We observe, first, that for any $\alpha > 0$ (so $p > 1$) the space $\mathscr{V}_0$ is reflexive. Given nondecreasing G_0 satisfying (5a,b,c) and Φ, f satisfying (13), the operator $T = L + A_0$ with A_0 given by (12) for $v \in \mathscr{V}_0$ satisfies the hypotheses of Browder's theorem quoted above by Lemma 3. Hence, the solution exists. If v_1, v_2 are two solutions of (14) so $Lv_1 + A_0(v_1) = 0 = Lv_2 + A_0(v_2)$, then applying the difference of these functionals to $(v_1 - v_2)$ gives

$$\begin{aligned} 0 &= \langle L(v_1 - v_2), v_1 - v_2 \rangle + \langle A_0(v_1) - A_0(v_2), v_1 - v_2 \rangle \\ &= B(\nabla v_1 + \Phi, \nabla v_2 + \Phi) \qquad (15) \\ &= \int_S \beta(s)\, \mu(ds) \end{aligned}$$

with $\beta(s)$ given for $[\nabla v_1 + \Phi](s)$, $[\nabla v_2 + \Phi](s)$ as in the proof of Lemma 2. As noted earlier, for G_0 nondecreasing, β is nonnegative so (15) can hold only if $\beta \equiv 0$; if G_0 is strictly increasing this means the arguments of B are equal at (almost) every $s \in S$ so $\nabla(v_1 - v_2) = 0$. Since v_1, v_2 vanish on $\mathbf{T} \times \partial\Omega$, it follows that $v_1 = v_2$ so the solution is unique. Suppose, now, G_0 satisfies (5d) as well. By Lemma 1, A_0 as defined by (12) is continuously dependent on $(\Phi, f) \in \mathscr{X} \times L_q(S)$. Suppose A_0 is defined by (12) for some particular $(\Phi_0, f_0) \in \mathscr{X} \times L_q(S)$ and v_0 is the corresponding solution of (10); so, $Lv_0 + A_0(v_0) = 0$.

By the continuity noted above, for any $\epsilon > 0$ we have

$$\| A_0(v_0) - A_1(v_0) \| < \epsilon$$

(where A_1 is similarly defined by (12) for some $(\Phi_1, f_1) \in \mathscr{X} \times L_q(S)$)

provided (Φ_1, f_1) is close enough to (Φ_0, f_0). Let v_1 be the solution of (10) corresponding to (Φ_1, f_1) so that $Lv_1 + A_1(v_1) = 0$. Then

$$\begin{aligned}\epsilon \| v_1 - v_0 \| &\geqslant [A_0(v_0) - A_1(v_0)](v_1 - v_0) \\ &= [\{Lv_0 + A_0(v_0)\} - \{Lv_0 + A_1(v_0)\}](v_1 - v_0) \\ &= [\{Lv_1 + A_1(v_1)\} - \{Lv_0 + A_1(v_0)\}](v_1 - v_0) \\ &= [A_1(v_1) - A_1(v_0)](v_1 - v_0)\end{aligned}$$

since $[Lv]v = 0$ for all v, in particular for $v = v_1 - v_0$. Thus,

$$\begin{aligned}\epsilon \| v_0 - v_1 \| &\geqslant [A_1(v_1) - A_1(v_0)](v_1 - v_0) \\ &= B(\nabla v_1 + \Phi_1, \nabla v_0 + \Phi_1).\end{aligned}$$

From (11) one has

$$\| v_1 - v_0 \|_{\mathscr{V}} \geqslant \| \nabla(v_1 - v_0)\|_{\mathscr{X}} = \|(\nabla v_1 + \Phi_1) - (\nabla v_0 + \Phi_1)\|_{\mathscr{X}}$$

so making ϵ small (i.e., letting $(\Phi_1, f_1) \to (\Phi_0, f_0)$) forces

$$\frac{B(\nabla v_1 + \Phi_1, \nabla v_0 + \Phi_1)}{\|(\nabla v_1 + \Phi_1) - (\nabla v_0 + \Phi_1)\|_{\mathscr{X}}} \to 0,$$

which, by Lemma 2, implies

$$\| \nabla(v_1 - v_0)\|_{\mathscr{X}} = \|(\nabla v_1 + \Phi_1) - (\nabla v_0 + \Phi_1)\|_{\mathscr{X}} \to 0.$$

By the Poincaré Lemma, $\| \nabla v \|_{\mathscr{X}}$ is equivalent to $\| v \|_{\mathscr{V}}$ for $v \in \mathscr{V}_0$ (as both v_1, v_0 are). Thus, $v_1 \to v_0$ in $\mathscr{V}_0$ as $(\Phi_1, f_1) \to (\Phi_0, f_0)$ and the solution of (10) has been shown continuously dependent on (Φ, f).

Remark 1. Returning to the original formulation (8), (9) of the problem, recall that $\Phi = \nabla\hat{\varphi}$, $f_1 = \hat{\varphi}^{\boldsymbol{\cdot}}$, where $\hat{\varphi}$ is a (t-periodic) extension to $\mathbb{R} \times \Omega$ of the original data φ, given on $\mathbb{R} \times \partial\Omega$. Standard extension theorems are available (cf., e.g., [4] or, for the case $p = 2$, the discussion in [7]) giving, subject to hypotheses regarding the smoothness of $\partial\Omega$, a space $\mathscr{F}$ of periodic functions on $\mathbb{R} \times \partial\Omega$ (i.e., functions on $\mathbf{T} \times \partial\Omega$) for which there is a continuous linear map:

$$\varphi \mapsto \hat{\varphi} \mapsto (\hat{\varphi}, f): \mathscr{F} \to \mathscr{V} \times L_q(S).$$

From Theorem 1 above, the solution map:

$$\varphi \mapsto (\hat{\varphi}, v) \mapsto u = v + \hat{\varphi}: \mathscr{F} \to \mathscr{V}$$

is continuous and the problem (8), (9) with periodicity conditions is well-posed.

Remark 2. The argument above is easily modified to apply to boundary conditions of other than the Dirichlet type already considered. The space $\mathscr{V}_0$ will then be the closure with respect to the norm (11) of the set of smooth (e.g., C^∞) functions satisfying the appropriate homogeneous boundary conditions. Observe that the particular nature of (9), (10b) above was of significance only in the repeated use of the Poincaré Lemma. It is only necessary to verify, then, that the norm (11) is dominated on the new $\mathscr{V}_0$ by the L_p-norm of the gradient. In particular, for the application indicated earlier to induced eddy currents it is appropriate to specify Neumann data for u. Since a potential is only significant modulo an additive constant, we impose, in addition, an auxiliary condition in defining $\mathscr{V}_0$ as the closure in the norm (11) of the smooth functions $v \in C^\infty(S)$ for which

$$\begin{cases} \nu \cdot \nabla v = 0 & x \in \partial\Omega \quad (\nu = \text{normal}), \\ \int_\Omega v\,dx = 0 & t \in \mathbf{T}. \end{cases}$$

It is easily seen that a Poincaré inequality holds for this $\mathscr{V}_0$.

Remark 3. No example is available for which the "static" formulation employed here leads to an existence proof not obtainable through use of the fixpoint argument of [3]. The approach seems of interest nevertheless, if only to provide an alternative viewpoint. Further, the continuous dependence seems to fit more naturally into such a setting.

4. Stability

Observe that in applications (e.g., in the case of the eddy current problem noted above) the scalar "coefficient" $\gamma(\cdot)$ is likely to be known only through measurement; it is as much a part of the "data" as φ is. Since γ is then given only approximately, it seems desirable to investigate the "structural stability" of the problem, i.e., continuous dependence of the solution on the function γ (or, equivalently, on G_0). This will also permit consideration of the dependence on the boundary data φ when perturbation of the period length is admitted.

THEOREM 2. *Let $\mathscr{G}$ be the class of maps G given as in* (3) *with each G_0 satisfying the conditions* (5a,b,d) *with the same α and the same $m(\cdot)$ in* (5d). *Impose on $\mathscr{G}$ the metric*

$$d(G, \tilde{G}) = \sup\{|G_0(r) - \tilde{G}_0(r)|/(1+r)^\alpha : r > 0\}. \tag{16}$$

Let $(\Phi, f) \in \mathscr{X} \times L_q(S)$ be fixed and let $A(v; G)$ be defined by (12) *for $G \in \mathscr{G}$. Then the solution of: $Lv + A(v; G) = 0$ depends continuously in $\mathscr{V}_0$ on $G \in \mathscr{G}$.*

Proof. This is quite similar to the proof of continuous dependence on (Φ, f). Suppose, for $G, \tilde{G} \in \mathscr{G}$,

$$| \tilde{G}_0(r) - G_0(r)| \leqslant \epsilon(1 + r)^\alpha, \qquad \text{for} \quad r > 0. \tag{17}$$

Then let the solutions using G, $\tilde{G}$ be v, $\tilde{v}$, respectively, and set $V = \nabla v + \Phi$, $\tilde{V} = \nabla \tilde{v} + \Phi$. One has

$$\begin{aligned}
B(V, \tilde{V}) &= [A(\tilde{v}; G) - A(v; G)](v - \tilde{v})\\
&= [\{L\tilde{v} + A(\tilde{v}; G)\} - \{Lv + A(v; G)\}](v - \tilde{v})\\
&= [\{L\tilde{v} + A(\tilde{v}; G)\} - \{L\tilde{v} + A(\tilde{v}; \tilde{G})\}](v - \tilde{v})\\
&= [A(\tilde{v}; G) - A(\tilde{v}; \tilde{G})](v - \tilde{v})\\
&= \int_S [G(\tilde{V}) - \tilde{G}(\tilde{V})] \cdot [V - \tilde{V}]\, \mu(ds)\\
&\leqslant \int_S | G_0(| \tilde{V} |) - \tilde{G}_0(| \tilde{V} |)| \cdot | V - \tilde{V} |\, \mu(ds)\\
&\leqslant \int_S \epsilon(1 + | \tilde{V} |)^\alpha\, | V - \tilde{V} |\, \mu(ds)\\
&\leqslant \epsilon[\mu(S)^{1/p} + \| \tilde{V} \|_{\mathscr{X}}]\, \| V - \tilde{V} \|_{\mathscr{X}}\\
&= C\epsilon \| V - \tilde{V} \|_{\mathscr{X}},
\end{aligned}$$

with C depending on $\tilde{V}$ (and, of course, on n, S, p) but not on V. It follows from Lemma 2 that $\| V - \tilde{V} \|_{\mathscr{X}} \to 0$ as $\epsilon \to 0$; since $V - \tilde{V} = \nabla(v - \tilde{v})$ and v, $\tilde{v}$ vanish on $\partial\Omega$, it follows from the Poincaré Lemma that $\| V - \tilde{V} \|_{\mathscr{X}}$ dominates $\| v - \tilde{v} \|_{\mathscr{V}_0}$. Thus, $v \to \tilde{v}$ in $\mathscr{V}_0$ as $G \to \tilde{G}$ in $\mathscr{G}$.

Remark 4. We note that this suffices to treat perturbation of the period length. In comparing a function w_τ having period τ with a function w_1 with period 1, as above, take the distance between w_τ and w_1 to be

$$| \tau - 1 | + \| \tilde{w}_\tau - w_1 \|, \tag{18}$$

where $\tilde{w}_\tau(s) = w_\tau(\tau s)$. Since, with this notation, $\partial \tilde{u}_\tau / \partial s = \tau\, \partial u_\tau / \partial s$, the equation

$$\partial u_\tau / \partial s = \nabla \cdot G(\nabla u_\tau) + f_0$$

must be replaced by

$$\partial\tilde{u}_\tau/\partial s = \nabla \cdot \tilde{G}(\nabla\tilde{u}_\tau) + \tilde{f}$$

with $\tilde{G} = \tau G, \tilde{f}(s) = \tau f_0(\tau s)$; the boundary condition is similarly modified—if $u_\tau = \varphi$ on $\mathbb{R} \times \partial\Omega$, then $\tilde{u}_\tau = \tilde{\varphi}$ with $\tilde{\varphi}(s) = \varphi(\tau s)$. Thus everything is referred to unit periodicity and it is easily checked that, for G satisfying (5a,b,d), $\tilde{G} \to G$ in $\mathscr{G}$ as $\tau \to 1$. With the topology indicated by the use of (18), we have shown that the solution of our problem depends continuously on φ, f even in the case of variable periodicity.

In the case $p = 2$ (i.e., G "essentially linear") it is possible to obtain —quite easily—a stability result as $t \to \infty$. Let u_1, u_2 be any two solutions (without regard to periodicity) of the differential equation (8) satisfying the same boundary conditions so $z = (u_1 - u_2)$ vanishes on $\mathbb{R} \times \partial\Omega$. Then, setting

$$N = N(t; z) = \int_\Omega |u_1(t, x) - u_2(t, x)|^2 \, dx,$$

one has

$$\begin{aligned} \dot{N} &= 2 \int_\Omega \dot{z}z \, dx \\ &= 2 \int_\Omega [\nabla \cdot G(\nabla u_1) - \nabla \cdot G(\nabla u_2)](u_1 - u_2) \, dx \\ &= -2 \int_\Omega [G(U_1) - G(U_2)] \cdot [U_1 - U_2] \, dx \\ &= -2B(U_1, U_2) \end{aligned}$$

with $U_1 = \nabla u_1$, $U_2 = \nabla u_2$ (each in $L_2(\Omega \to \mathbb{R}^n)$) and B defined here with $S = \Omega$. Now the estimate in Lemma 2 shows that for G_0 satisfying (5d) with $\alpha = 1$ one has

$$B(U_1, U_2) \geqslant \psi_0(\| U_1 - U_2 \|_{\mathscr{X}})$$

with

$$\psi_0(\delta) := C_2 m(\epsilon_1/4)\, \delta^2,$$
$$\epsilon_1 = \epsilon_1(\delta) := \delta/2(2\mu(\Omega))^{1/2}$$

(noting that we may set $\epsilon_2 = 0$ for $p \geqslant 2$). Observe that $m(\cdot)$ may be taken to be nondecreasing in (5d). Since $(u_1 - u_2)$ vanishes on $\partial\Omega$, it follows from the Poincaré Lemma that $C^2 \| U_1 - U_2 \|_{\mathscr{X}}^2 \geqslant N$ for some fixed constant C whence

$$\dot{N} \leqslant -\psi(N)$$

with $\psi(\delta^2) := 2\psi_0(C\delta)$. Integrating gives

$$\Psi(N(t)) \leqslant -t \to -\infty \qquad \text{as} \quad t \to \infty$$

with

$$\Psi(\nu) := -\int_\nu^{N(0)} ds/\psi(s).$$

Since ψ is bounded away from zero on $[\epsilon, \infty]$ for any $\epsilon > 0$, it is clear that $\Psi(\nu) \to -\infty$ implies $\nu \to 0$. We state this as a theorem.

THEOREM 3. *Let G be given as in* (3) *with G_0 satisfying* (5d) *with $\alpha = 1$. Let u_1, u_2 be two* (*weak*) *solutions of the diffusion equation* (8) *with the same boundary data* (*i.e., $u_1 = u_2$ on $\mathbb{R} \times \partial\Omega$*). *Then*

$$N(t) := \int_\Omega [u_1(t, x) - u_2(t, x)]^2 \, dx \leqslant \Psi^{-1}(-t) \tag{19}$$

where

$$\Psi(\nu) = -\int_\nu^{\nu_0} ds/\psi(s),$$

$$\psi(s) = C_0 m(C_1 s^{1/2}) s, \qquad \nu_0 = N(0),$$

with the constants C_0, C_1 depending only on Ω. Thus, $N(t) \to 0$ as $t \to \infty$. In particular, if the boundary data is periodic and there is a periodic solution u_1 then it is asymptotically stable.

5. Further Remarks

It is interesting to observe that A, as defined in (12), is the gradient of a scalar functional.

Setting

$$\Gamma(r) = \int_0^r G_0(\rho) \, d\rho, \qquad \text{for} \quad r > 0,$$

we define the functional

$$g[v] = \int_S [\Gamma(|\nabla v(s) + \Phi(s)|) + fv(s)] \, \mu(ds). \tag{20}$$

A simple formal calculation shows that A is the Gateaux derivative of g; since A has been shown continuous on $\mathscr{V}$ for G satisfying (5a,b), it follows that A is the Fréchet derivative of g. It is clear that g will be convex (hence, A monotone) for G_0 nondecreasing and that g will be strictly convex for G_0 strictly increasing. Lemma 2 provides, for G_0 satisfying (5d) as well, a strict convexity estimate adequate to obtain the continuity result desired.

Throughout the formulation above it was (implicitly) assumed that G was independent of (t, x). It should be clear that this assumption is entirely inessential and was made for simplicity only. If $G = G(t, x; \cdot)$ then the proofs remain essentially unchanged if, in condition (5), it is assumed that $\alpha, M_0, m_0, m(\cdot)$ can each be taken to be independent of (t, x). Treatment of equations involving any dependence of G on the value of u, however, would require much more extensive modification of the arguments.

Another minor generalization is also easily available. Replace (3) by

$$G(X) = \gamma(|\mathbf{A}X|)\,\mathbf{A}^*\mathbf{A}X, \qquad \text{for} \quad X \in \mathbb{R}^n \tag{21}$$

with $\mathbf{A} = \mathbf{A}(t, x)$ an $n \times n$ matrix for $(t, x) \in \mathbf{T} \times \Omega$ such that the eigenvalues of $\mathbf{A}^*\mathbf{A}$ are uniformly bounded away from 0, ∞. Then the proofs of Lemmas 1, 2 may be simply modified by substitution of $\mathbf{A}X$ for X; observe that the uniform bounds on the eigenvalues of $\mathbf{A}^*\mathbf{A}$ ensures that the norms

$$\left[\int_S |X|^p \mu(ds)\right]^{1/p}, \qquad \left[\int_S |\mathbf{A}X|^p \mu(ds)\right]^{1/p}$$

are equivalent.

Finally, we state without further proof an abstract version of Theorem 1.

THEOREM 4. *Let $\mathscr{W}_0$ be a closed subspace of a reflexive Banach space $\mathscr{W}$. Let ψ: $\mathbb{R}^+ \to \mathbb{R}^+$ be strictly increasing and unbounded. Suppose for each $t \in \mathbf{T}$ the map $T = T(t, \cdot)$: $\mathscr{W} \to \mathscr{W}_0{}^*$ is continuous on $\mathscr{W}$ and satisfies, for some $p > 1$,*

$$[T(t, w_1) - T(t, w_2)](w_1 - w_2) \geqslant C(\epsilon)\,\psi(\|w_1 - w_2\|)\,\|w_1 - w_2\|^p, \tag{22}$$

for $w_1, w_2 \in \mathscr{W}$ whenever $(w_1 - w_2) \in \mathscr{W}_0$ and

$$\max\{\|w_1\|, \|w_2\|\} \geqslant \epsilon\,\|w_1 - w_2\|.$$

Let $\mathscr{V} = L_p(\mathbf{T} \to \mathscr{W})$, $\mathscr{V}_0 = L_p(\mathbf{T} \to \mathscr{W}_0)$; observe that $\mathscr{V}_0$ is reflexive with $\mathscr{V}_0{}^*$ embedded in $\mathscr{V}^* = L_q(\mathbf{T} \to \mathscr{W}^*)(1/p + 1/q = 1)$. Let L: $\mathscr{D} \to \mathscr{V}_0{}^*$ be the (skew-adjoint) closed extension of $\partial/\partial t$: $C^\infty(\mathbf{T}) \to \mathscr{V}_0{}^*$).

Then for each pair $(w, f) \in \mathscr{V} \times \mathscr{V}_0{}^*$ there exists a unique (weak) solution v of

$$Lv + T(v + w) = f, \tag{23}$$

i.e., there exists $v \in \mathscr{D}$ *such that*

$$0 = \int_T [\dot{v} + T(t, v + w) - f(t, \cdot)]u \, dt \qquad \textit{for} \quad u \in \mathscr{V}_0 .$$

Further, this solution v *depends continuously on* (w, f).

The proof of this parallels that of Theorem 1.

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